

Maths Circle Explorations: Session 8

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Problem 1

A few questions about finding patterns in colourings:

1. Suppose we colour all the integers with two colours, red and blue. Verify that regardless of how we choose the colouring, we always find three integers which are the same colour and are evenly spaced? (By “evenly spaced”, we mean that the integers should be of the form a , $a + d$ and $a + 2d$ where d is a positive integer.)
2. We say that n numbers a_1, a_2, \dots, a_n are in arithmetic progression if they are of the form $a, a + d, \dots, a + (n - 1)d$. So, part (1) is about finding arithmetic progressions of length 3, such that all the terms have the same colour, i.e. they are monochromatic. Is it possible to find longer arithmetic progressions which are monochromatic? Or is it possible to colour the integers in such a way that monochromatic arithmetic progressions of a certain length are avoided?
3. Is it possible to find monochromatic progressions of infinite length?
4. What happens if we increase the number of colours?
5. The integers can be visualized as points on a line. So, the above questions refer to colouring points on a line. What if we colour points in a plane or in three-dimensional space? What kind of monochromatic patterns can we hope to find (or not find)? For example, can we always find a monochromatic equilateral triangle? What about other patterns?