

Maths Circle Explorations: Session 2

November 12, 2021

2. Recall the two-player game from Activity 5 of Session 1. Perumal and Shehnaaz have been playing this game, in search of patterns. Based on their observations, Shehnaaz speculates that the initial configurations (n_1, n_2, n_3) of boxes can be organized into two types, L and W, such that
- (a) Each configuration is of exactly one of the two types.
 - (b) $(0, 0, 0)$ is of type L.
 - (c) If a configuration is of type L then any legal move will convert it into a configuration of type W.
 - (d) If a configuration is of type W, then there is at least one legal move that converts it into a configuration of type L.

She claims that if the initial configuration is of type L, then the second player has a winning strategy.

Perumal has recently learned that for each natural number n , there is exactly one way to write it in the form

$$n = a_0 + 2a_1 + 2^2a_2 + \dots + 2^ka_k \quad (1)$$

with a_0, a_1, \dots, a_k taking the values 0 or 1 (why is this true?). For example

$$11 = 1 + 2(1) + 2^2(0) + 2^3(1).$$

Through careful experiment Perumal and Shehnaaz have discovered that certain configurations (n_1, n_2, n_3) are losing positions for the first player. In a moment of inspiration, Perumal writes down the numbers (n_1, n_2, n_3) in these configurations in the form given in Equation 1. Staring at the table he has created, he is surprised to find a clear pattern emerge! He describes a rule that organizes the initial configurations into two types L and W.

- (i) What is the rule that Perumal described?
- (ii) Perumal and Shehnaaz are able to check that this rule does satisfy Shehnaaz's criteria (a) – (d) above for small values of (n_1, n_2, n_3) . However, they are unable to convince themselves that this rule works for very large numbers of boxes (large values of the numbers n_i). Do you believe the rule is correct? Can you convince them of your belief?