Maths Circle India: Module 9, Session 36
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1 A simple card game

We start with the following:

> **Problem**: Two players A and B have a pack of 9 cards. They take turns to remove $1, 2$ or 3 cards from the pack–but a player must never remove the same number of cards as the previous player. The winner is the one who either takes the last card or leaves the other player with no valid move. Who wins?

Note that either the pack goes on getting smaller until someone takes the last card, or there are some cards left but a player has no valid move. The first player can choose one of three moves and start the game. For simplicity assume that A is the first player. Can A choose a first move that will guarantee a win. Or can player B somehow regain control by always choosing the 'correct' response? Now try as many exercises as possible from the list below and try to solve the problem above.

1. If the pack has exactly 1 card, then A will remove it and win the game. Verify that if there are 4 cards then irrespective of what A does, B will always win. Play and find out who can force a win if A always starts and the number of cards in the pack ranges from 1 to 8. Make a table.

2. Now back to the original problem of 9 cards. Suppose A starts by removing 3 cards. What should B do to make sure of the winning.

3. Suppose A starts by removing 2 cards. Can B still make sure of winning? Explain.

4. Suppose A starts by removing 1 card. Can B make sure of winning? Explain.

5. Which player loses the game with 8 cards? Will the same player loose if the first player is only allowed to take 2 or 3 cards on his first move?

6. Prepare a table similar to the one above for a pack of 9 cards up to 15 cards.

7. Who would you expect to win when the pack starts with 16 cards? What should their first move be?

8. Two players take turns to remove either 1 or 3 cards from a starting pack. The winner is the one who removes the last card. If the opponents move can't be repeated, who can for a win? How? (you may want to try out various small values of number of cards in the pack).

2 More games

Here are some other problems you can try. Instead of attempting a problem in the first go, try to break it down into smaller, leading problems (as we did above) and try to work on them.

> **Problem 2**: The game starts with two piles of matches. Two players take turns to remove either any number of matches from any one pile, or equal numbers of matches from both piles. The winner is the one who takes the last match. When can the first player force a win? How should he play in order to win?

Let the pair (m, n) denote the number of matches in the pile. We can keep track of state of affairs with such ordered pairs denoting the numbers of matches remaining in each pile after a move has been made. Let the pair (a, b) denote some state; the next move changes it to exactly one of the following pairs: $(a - t, b)$, or $(a, b - t)$, or $(a - t, b - t)$ for some $t > 1$.

Here is an example; starting piles are $(18, 15)$. First A withdraws 2 each to leave $(16, 13)$. Then B plays to $(12, 9)$; A plays to $(5, 9)$; B plays to $(2, 6)$; A plays to $(2, 1)$. At this B four options $(1, 1), (0, 1), (2, 0)$ or $(1, 0)$. Whichever option B chooses A will always pick up the last match and win the game. You can try various combinations and check whether A always wins.

> **Problem 3**: Start with a pair of positive integers, say $(a, b), a \leq b$. The two players then toss to decide who goes first. The first player changes (a, b) by subtracting any multiple of the smallest number from the larger to produce a new pair. Negative numbers are forbidden. The second player then transforms this new pair in the same way, and so on. The first player to produce a pair in which one of the numbers 0 is the winner.

Try this game for various pairs of small positive integers. When can the first player force a win? How should he play in order to win? Does this game remind you of any mathematical algorithm? Explain.