TIFR-STCS Maths Circle Team

Session 1: April 15, 2022

4 Vote for picnic!

Ravi, Kiran and Hasan are planning to go to a picnic in their winter vacation. They have three places to choose from - Lonavala, Juhu and Uran. Everyone has a preference list as follows:

- Ravi's list - Juhu $>$ Lonavala $>$ Uran
- Kiran's list - Lonavala $>$ Uran $>$ Juhu
- Hasan's list - Uran $>$ Lonavala $>$ Juhu

Now, they use the following rule to decide on the place to go:

1. Between Lonavala and Juhu, two people (Kiran and Hasan) prefer Lonavala over Juhu. So Lonavala wins over Juhu.
2. Between Lonavala and Uran, two people (Ravi and Kiran) prefer Lonavala over Uran. So Lonavala wins over Uran.

Lonavala wins over the other two places so they decide to go to Lonavala for their picnic.

They now wonder if they could have rearranged their lists in such a way that there would have been no winning place. Do you think they can do so? Can you help them come up with such a set oflists?

Their picnic now done with and their vacations now over, Ravi, Kiran and Hasan are now back at school. After a month, the school decides to organize an educational tour. Now, of course, there are many more students involved in addition to Ravi, Kiran, and Hasan, so let us call the students $s_1, s_2, ..., s_n$, where n is the number of students. The number of options for places to visit is also much larger, so let us call these places $p_1, p_2, ..., p_m$, where m is the number of places.

Interestingly, Hasan's class-teacher comes up with the same rules that Ravi, Kiran and Hasan had used earlier to decide their picnic spot: every student will give a ranking of the places, and a place that wins over every other place will be chosen.

Needless to say, Hasan is alarmed at this turn of events. Using the examples that he had constructed earlier in collaboration with Ravi and Kiran, he convinces the teacher that this might not work out. The teacher agrees, and after thinking for some time, modifies the rules as follows:

The teacher prepares her ownranking of the places, which happens to be $p_1 > p_2 > p_3 > \ldots p_m$. She then lays down the rule that the preference lists for the students have to be of a specific form. This form is described next. Each student s_k can choose one of the places, say p_i, as his/her top preference. However once the student s_k has chosen his/her top choice p_i, the rest of his/her preference list must look like the following:

- It must first list down, in the same order as the teacher's list, all the places $p_i, p_{i+1}, p_{i+2} \ldots, p_m$.
- Then, it must list down all the remaining places in reverseorder of the teacher's list, i.e, in the order $p_{i-1}, p_{i-2}, \ldots, p_1$.

Thus, the if p_i is the top choice of student s_k, then her preference list will look like

$$p_i >_k p_{i+1} >_k \ldots >_k p_m >_k p_{i-1} >_k p_{i-2} >_k \ldots >_k p_1.$$

Here the k in the subscript in the notation $>_k$ denotes that this is the preference list of student s_k.

As before, we say that place p_i wins over p_j if the number of students who prefer p_i over p_j is greater than the number of students who prefer p_j over p_i. And again, the teacher will choose the place which wins over all other places.

Assume that the number n of students is odd. The teacher claims that now, no matter how the students prepare their lists, as long as they follow her rules, there will always be a single winning place.

Is she right? Why?