Math-Circle: Session 3

TIFR-CAM and ICTS

November 21, 2022

Problem 2. Consider arithmetic progressions

$$A_1 = \{a_1, a_1 + d_1, a_1 + 2d_1, \ldots\},$$

$$A_2 = \{a_2, a_2 + d_2, a_2 + 2d_2, \ldots\},$$

$$A_3 = \{a_3, a_3 + d_3, a_3 + 2d_3, \ldots\},$$

$$\vdots$$

$$A_m = \{a_m, a_m + d_m, a_m + 2d_m, \ldots\},$$

where for each A_k, a_k is the first term and d_k is the common difference. Assume that arithmetic progressions $A_1, A_2, ..A_m$ are pairwise disjoint. Also, assume that $\mathbb{N} = \bigcup_{k=1}^{m} A_k$. Prove that

- $\frac{1}{d_1} + \frac{1}{d_2} + .. + \frac{1}{d_m} = 1$
- For some $i \neq j$, $d_i = d_j$.
- (Bonus problem) Can you find all such arithemetic progressions $A_1, A_2, ..A_m$ such that the above conditions hold?

Hint: Given $N = \bigcup_{k=1}^{m} A_k$ and A_k are pairwise disjoint, the first natural conclusion is that size of N must be equal to sum of sizes of A_k. If your notion of size of a set is the number of elements in that set (i.e. its cardinality), you will get $\infty = \infty + \infty + \infty + \ldots + \infty$. This does not give you any useful information. Recall however that you have a freedom in choosing the notion of size. This is done by choosing any numbers a_n of your choice as explained above. Note that the standard notion of size using cardinality corresponds to $a_n = 1$ for all n. Think of some other choice of numbers a_n so that sum $\sum_{n \in A_k} a_n$ can be computed in terms of a simple formula. Then, use the conclusion that the size of $\mathbb{N}$ must be equal to sum of sizes of A_k to derive information about arithmetic progressions A_k.