Math-Circle: Session 3

TIFR-CAM and ICTS

November 21, 2022

Problem 3. Let R be a rectangle. Suppose you tile the rectangle R using smaller rectangles $R_1, R_2, .., R_m$. The sides of each rectangle R_k is parallel to the corresponding sides of R. The dimensions of rectangles R_k, i.e. lengths of its sides, may be different from each other (here, by length I mean the standard length in SI unit). Suppose that each rectangle R_k has at least one of its side (either its length or its breadth or both) of integer standard length in SI unit. Prove that the bigger rectangle R also has at least one of its side of integer standard length in SI unit.

Follow the following steps to solve the above problem:

- If $[a, b]$ is an interval on the real line, the standard notion of length gives the length of $[a, b]$ to be $b - a$. But, with the generalized notion of length as described above, for any function F (e.g. $F(x) = x^2, e^x, \sin(x)$ or any other function of your choice), you can assign to interval $[a, b]$ the number $F(b) - F(a)$ as its length. That is, we are defining $\mu([a, b]) = F(b) - F(a)$. Check that this definition indeed defines a measure μ which satisfies (0.1).

- If R is a rectangle with its vertices at $(a, x), (b, x), (b, y), (a, y)$ in the coordinate plane, the standard notion of area gives the area of this rectangle as $(b - a)(y - x)$. But, with the generalized notion of area as described above, for any function F, you can assign to the rectangle R the number $(F(b) - F(a)) \times (F(y) - F(x))$ as its area. That is, we are defining $\mu(R) = (F(b) - F(a)) \times (F(y) - F(x))$. Check that this definition indeed defines a measure μ which satisfies (0.1).

- Now lets go back to the framework of the above problem. If a rectangle R is tiled using smaller rectangles $R_1, R_2, ..R_m$, the most immediate conclusion is that the area of R is sum of areas of R_k. That is,

$$\mu(R) = \mu(R_1) + \mu(R_2) + .. + \mu(R_m). \tag{0.2}$$

  If you work with the standard notion of area which is $\mu(R) = (b - a)(y - x)$, the equation (0.2) may not give you any information. But, note that you also have a freedom of the function F while defining the notion of area. Try to think of a function F so that you can derive information about the rectangle R using the equation (0.2). While thinking of such a function F, try to use the given information that at least on side of each rectangle R_k is of integer length.