Math-Circle: Session 4

TIFR-CAM and ICTS

December 1, 2022

1 Planets, rationals and clocks

In this story we will learn the relationship between three seemingly disparate objects: planets, rationals and clocks. Suppose you were tasked to build a clock. You have a motor which turns around at a certain speed. Can you use it to build a clock? What you need for this are gears (see Figure 1)? A gear is a circular disk with uniformly spaced teeth at its boundary. Now notice that if you have two gears of different radii attached to each other, they rotate at different rates. Our first task will be to understand how exactly. By the way one can find gears in the natural world as well. Look at the following link.)

(a) Gears on bicycles: Most modern conveyances have gears: cycles, cars and so on. Cycles have in fact two kinds of gears. One of them is attached to the pedal in the front and one on the back attached to the back. Both in the front and the back there is a choice of various radii for the gears (see Figure 2). There is a chain connecting the back and the front set of gears. Each time you pedal the chain goes around the front gear a certain amount (according to the front radii) and around the back gear a certain propotional amount. The angle of rotation of the back gear determines the distance that you move forward.

Determine whether you would like a smaller radii gear or a bigger radii gear in the front and the back in the following situations. What is the preference going uphill (remember that you would prefer to move lesser with each time you pedal)?

(b) Now suppose two gears are next to each other as in Figure1. The first gear has 50 teeth and the other one has 27 teeth. Suppose the first gear starts rotating. After how many complete rotations of the first gear will the second gear also have rotated an integer number of rotations.

(c) Clocks and gear configurations: Clock gears are a little more complicated. Here you want two different hands (or three if you want to keep track of the second's hand as

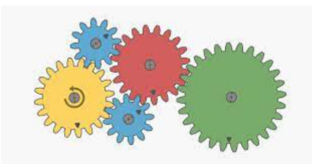

Figure 1: Gears

Figure 2: Gears attached to the back wheel

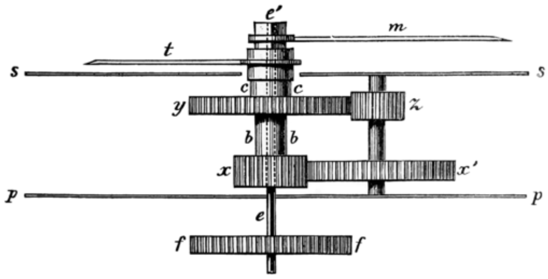

Figure 3: Gears of a watch

well) which will rotate at different rates. Look at Figure 3. There are two hands m and t and five gears f, x, x', z and y. Here f is attached to the motor and its rate of rotation determines the movement of t. The rate of rotation of x is the same as that of f. The rate of rotation of x' is determined by the rate of rotation of x and the number of teeth that x and x' have. z rotates at the same rate as x'. The rate of rotation of y is determined by the rate of rotation of z and the number of teeth that y and z have. Finally the rates of rotation of y and t are the same. Suppose x, x', z and y have n_1, n_2, n_3 and n_4 teeth. What is the ratio of the rate of rotation of m and t? In particular show that the ratio of the rate of rotation is rational.

Now you can imagine that a very similar construction can be made for the solar systems where the planets move at different rates around the sun. However for any such construction the ratio of the rate of rotation of the various planets has to be rational. This need not be true always. So here arises the main mathematical question:

(d) Approximations of irrationals: Prove that you can approximate irrationals by rationals. For example, the rational approximations of π are 3.14, 3.141, 3.1415,... and so on. Show that any irrational number x can be approximated by rational numbers.

(e) Best approximations of irrationals: Now the question is whether you can approximate these irrationals well. There are "best approximations" but finding them is not easy and need you to study a little more. Here is an example which is easier. For all natural numbers Q, can you find natural numbers p, q where $q \leq Q$ such that

$$|x - \frac{p}{q}| \leq \frac{1}{Q^2}?$$

Next, prove that there are infinitely many rational numbers $\frac{p}{q}$ with $\gcd(p, q) = 1$ such that $|x - \frac{p}{q}| < \frac{1}{q^2}$.

You can follow the following steps to prove the above.

- Prove that there exists a rational number $\frac{p}{q}$ such that $|x - \frac{p}{q}| < \frac{1}{2q}$.
- For each integer $n \geq 1$, there exists a rational number $\frac{p}{q}$ with $1 \leq q \leq n$ and $|x - \frac{p}{q}| < \frac{1}{nq}$.