Maths Circle India: Module 9, Session 33
Organized by Chennai Mathematical Insitute

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1 Warm up

By an arrangement of lines we mean a finite collection A of $n \geq 1$ different lines $\{l_1, \ldots, l_n\}$ in the plane. The goal of today's session is to explore various (combinatorial) questions associate with line arrangements. For example, one can try and find a formula for the number of pieces these n lines cut the plane into. Note that one line cuts the plane into exactly 2 pieces. *Can you write a formal proof of this fact?* For this particular question, it doesn't matter exactly which line is chosen; any line, when removed, will cut the plane into exactly 2 pieces. (Here lines are uniquely identified by their defining equations)

Your first assignment of this session is to define a reasonable notion of **equivalence of line arrangements**. The following easy observations should help you do that. If two arrangements have different number of lines then they are certainly not equivalent. Any two arrangements of one line each are equivalent. However, when you consider two lines there is more than one way to arrange them (up to equivalence). In Figure 1 you will see three non-equivalent arrangements of 3 lines. There are two arrangements which cut the plane into the same number of pieces, however, we would not want them to be equivalent.

For $n = 1, 2, 3, 4$ draw arrangements of n lines, up to equivalence. If you are up to it, you can also try 5 lines; that number is fairly large so don't spend too much time.

2 How many pieces?

Suppose A is an arrangement of n lines, denote by $f(A)$ the number of pieces of the plane cut by these n lines in A.

1. For every n what is the minimum possible value of $f(A)$? Is there anything special about A in this case?

2. For $n = 1, 2, 3, 4$ find all possible values of $f(A)$.

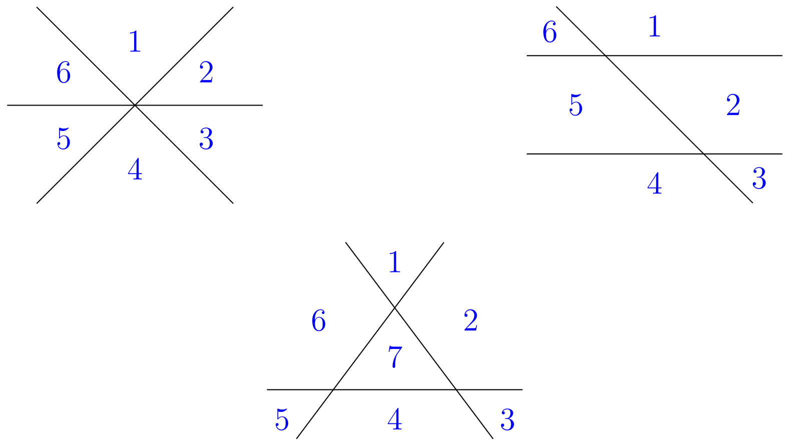

Figure 1: Some arrangements with $n = 3$ lines.

3. Find a formula, in terms of n, for the maximum number of pieces cut by any arrangement of n lines. (Hint: try to have as many intersections as possible and then use recurrence.)

4. For a given n how would you arrange n lines so that the number of pieces is exactly $2n$?

5. Is is possible to arrange $n \geq 3$ lines so that the number of pieces is $n + 2$? Justify your answer.

Note: If possible, please scan your solutions (drawings, calculations, observation tables etc.) and keep the file ready during the session.