Maths Circle India: Module 9, Session 34
Organized by Chennai Mathematical Insitute

21 April 2023

1 Sums of Squares

The goal of today's section is to explore various questions in number theory associated with perfect squares and their sums. Here, by perfect squares we mean squares of natural numbers or 0. Tautologically, perfect squares are positive integers that can be written as a sum of 1 perfect square. Your first assignment of this session is to find out which are the positive integers that can be written as a **sum of two perfect squares**. To this end, find out, for each positive integer n up to 200, if it can be written as a sum of two perfect squares. If you can, extend the same search for integers up to 1000. Can you find any patterns out of this? Record your observations and make conjectures if possible.

Are there any numbers that cannot be written as sums of two squares? If so, the next assignment of this session is to find out which are the positive integers that can be written as a **sum of three perfect squares**. To this end, find out, for each positive integer n up to 200, if it can be written as a sum of three perfect squares. If you can, extend the same search for integers up to 1000. You should also explore if the positive integers that can be written as a sum of two perfect squares can also be written as a sum of three perfect squares without using 0 as one of the perfect squares. Can you find any patterns out of this? Record your observations and make conjectures if possible.

Are there any numbers that cannot be written as sums of three squares? If so, the next assignment of this session is to find out which are the positive integers that can be written as a **sum of four perfect squares**. To this end, find out, for each positive integer n up to 200, if it can be written as a sum of four perfect squares. If you can, extend the same search for integers up to 1000. You should also explore if the positive integers that can be written as a sum of two perfect squares can also be written as a sum of three perfect squares without using 0 as one of the perfect squares. Can you find any patterns out of this? Record your observations and make conjectures if possible.

We should also ask ourselves at this point the following question - suppose

it is possible to write some integer n as a sum of two, three, or four perfect squares. Then in how many (distinct) ways can you write n as a sum of two, three, or four perfect squares? Call these numbers $r_2(n)$, $r_3(n)$, and $r_4(n)$ respectively. What, if anything, can you say about these $r_2(n)$, $r_3(n)$, and $r_4(n)$? Record your observations and make conjectures if possible.

You should also find out if we can omit 0 from the list of our perfect squares, and if that changes anything. Can you make similar explorations for 2, 3, 4, 5 perfect squares in this case?