Maths Circle India: Module 9, Session 35
Organized by Chennai Mathematical Institute

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Wiring diagrams

A wiring diagram with n wires is built by stacking blocks on top of each other. A *block* consists of n wire-segments with 2 adjacent wires crossing. There are two types of blocks for $n = 3$ which are shown below.

Labeling the wires $1, 2, \ldots, n$ at the bottom and following where they end up, we obtain a permutation/rearrangement[1] of these numbers as illustrated in Figure 1 for $n = 3$.

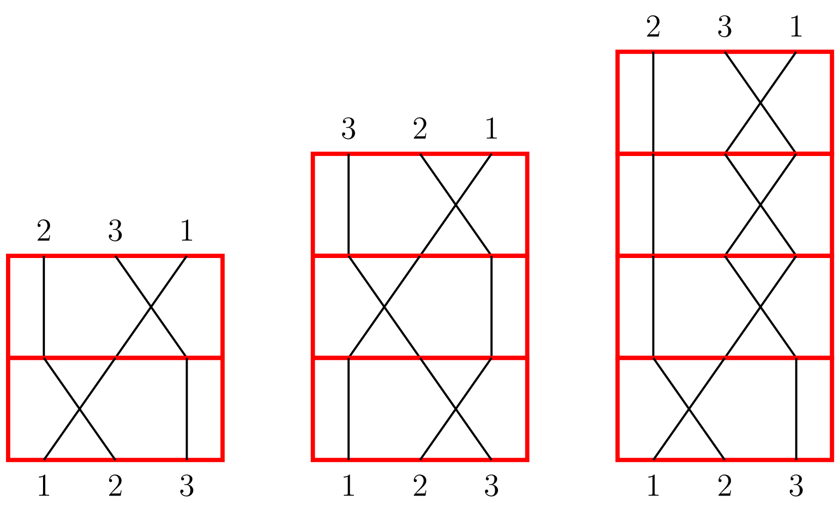


Figure 1: Some wiring diagrams with $n = 3$ wires.

[1] A rearrangement of the numbers $1, 2, \ldots, n$ is called a permutation of size n. There are 6 permutations of size $n = 3$, which are:

1 2 3   1 3 2   2 1 3   2 3 1   3 1 2   3 2 1

In general, there are $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ permutations of size n (try to prove this if you haven't seen this before).

Your first task is to play around and come up with various wiring diagrams with $n = 3$ wires that give the permutation 3 2 1; one example is the second wiring diagram in Figure 1. While you're doing this, think about the following questions:

- How many such wiring diagrams are there?
- Is there a restriction on the number of blocks such a wiring diagram can have?
- What is the minimum number of blocks required to make such a wiring diagram?

Try similar experiments for other permutations as well.

Removing redundancies

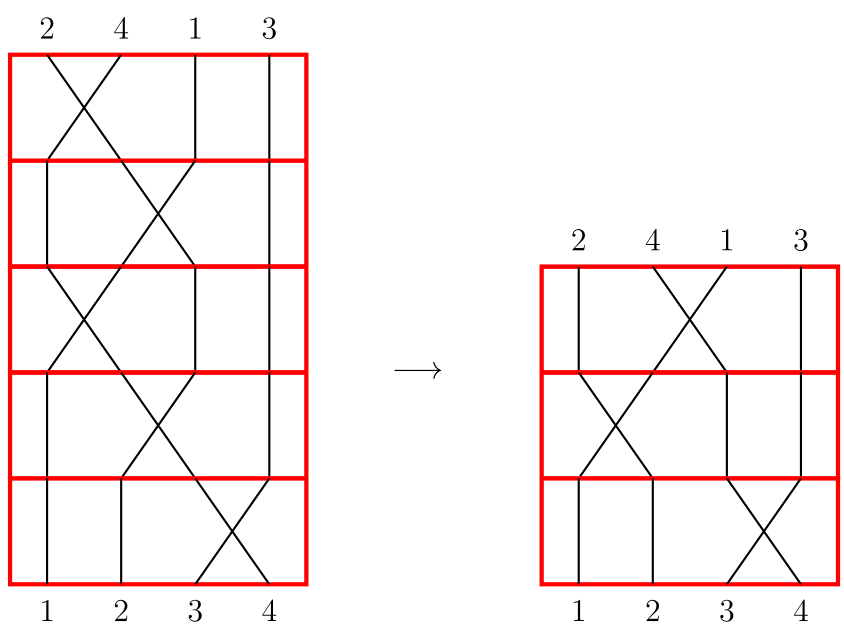
In a wiring diagram, a pair of wires $\{i, j\}$ is said to be odd (respectively even) if wire i and wire j cross an odd (respectively even) number of times. For example, in the last wiring diagram in Figure 1, the pairs $\{1, 2\}$ and $\{1, 3\}$ are odd and the pair $\{2, 3\}$ is even. Note that the same is true for the first wiring diagram in Figure 1.

- In fact, if two wiring diagrams give the same permutation, then the even pairs and odd pairs of wires in both diagrams are the same. That is, the parity of a pair of wires only depends on the permutation.

  Can you tell just by looking at the permutation which pairs of wires are odd and which are even? Justify.

- A wiring diagram is called *optimal* if the following holds: Any odd pair of wires cross each other exactly once and any even pair of wires do not cross each other.

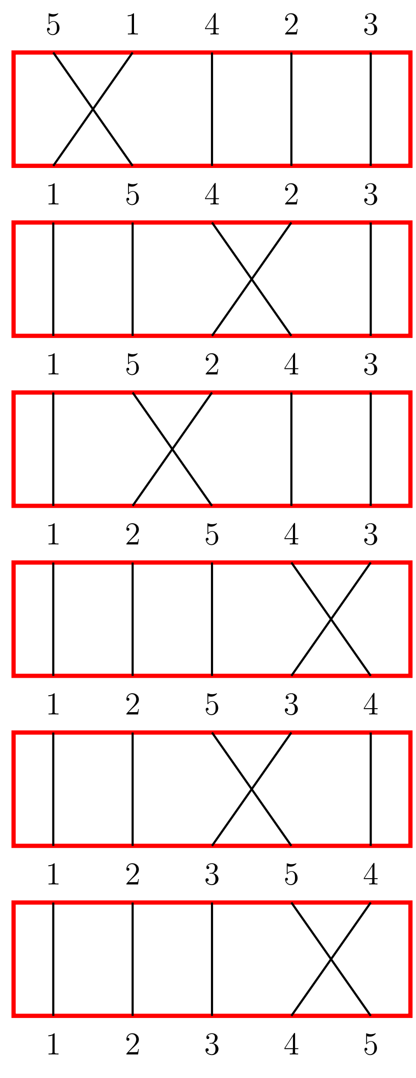
  Show that we can delete blocks from a wiring diagram in such a way that we end up with an optimal wiring diagram that gives the same permutation. A clue as to how to do this is in the following diagram. Consider the pair of wires $\{2, 4\}$.



- Suppose we are given a permutation that is obtained from a wiring diagram. Determine the minimum number of blocks in a wiring diagram that gives us this permutation.

Building wiring diagrams

We will now show that any permutation of size n can be obtained from a wiring diagram with n wires. To do this, we read the wiring diagrams from top to bottom and note that each block just swaps adjacent terms in a permutation. For example, if we are given the permutation 5 1 4 2 3 of size 5, consider the following:



Can you come up with a general method to obtain a wiring diagram that results in a given permutation?

Note that this problem is equivalent to the following: You are given n cards labelled $1, 2, \ldots, n$ that are placed in a row in some order. You are allowed to swap any two adjacent cards. Can you perform such swaps so that you end up with the cards in increasing order $1, 2, \ldots, n$?

Food for thought

We now consider two interesting questions related to the concepts we have seen.

- Among all the wiring diagrams with $n = 3$ wires that give the the permutation 3 2 1, how many have 3 blocks? How many have $4, 5, 6, \ldots$ blocks?

  In general, given a permutation of size n, how many wiring diagrams with k blocks give us this permutation? There is probably no simple answer to this question. Experiment with various/special permutations and try to find patterns!

- Can you think of rules that allow you to 'transform' wiring diagrams in such a way that you end up with the same permutation? For example, the diagram below gives an idea for one such rule: