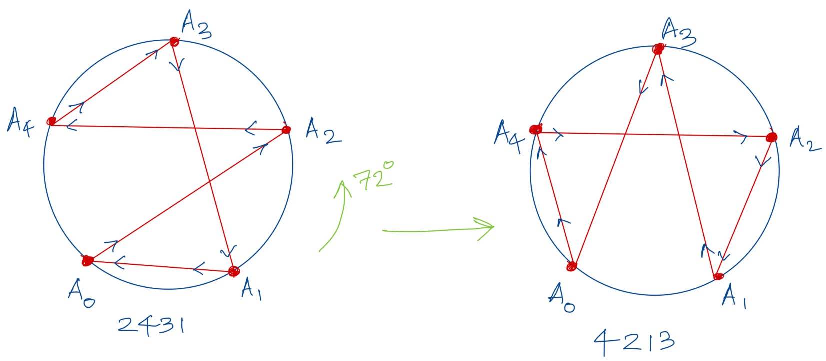
MCI Module 10, Session 37
Conducted by IISER Bhopal, June 10, 2023

Primes, permutations and a magic formula

1. In her maths class, Aakriti learns that for a natural number n, the number $n!$, called the *factorial* of n is defined as $n! = n(n-1)(n-2)(n-3)\cdots 2(1)$. For example, $4! = 4(3)(2)(1) = 24$. She learns that $n!$ is the total number of distinct orderings of integers $1, 2, \ldots, n$. Each such ordering is called a *permutation* of the the set of first n natural numbers. A permutation is represented by $j_1 j_2 \ldots j_n$, where j_1 indicates the number that replaces 1 in the new ordering, j_2 represents the number that replaces 2 in the new ordering, and so on. For example, for $n = 3$ we have $3! = 6$ permutationse$123, 132, 213, 231, 312$ and 321. She also learns that a natural number $n > 1$ is a *prime number* if 1 and n are its only divisors. For example, 5 is a prime number but since $6 = 2(3)$, it is not a prime number. Aakriti's teacher tells her that there is a connection between factorials and prime numbers and asks her to find it.

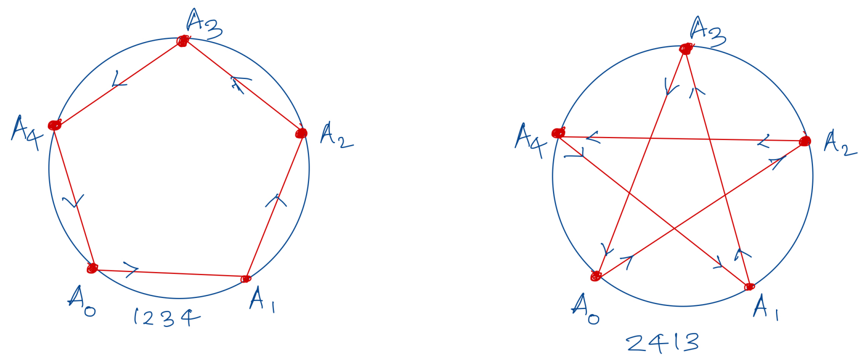
   (a) Given a prime number p, for each permutation of $\{1, 2, 3, \ldots, (p-1)\}$, Aakriti draws a corresponding closed figure as follows. She first draws a circle. For $p = 5$, for example, she considers five points which are the vertices of a regular pentagon inscribed inside this circle. Suppose the vertices are labeled as A_0, A_1, A_2, A_3, A_4. Then to represent the permutation 2431, she joins A_0 to A_2, then A_2 to A_4, then A_4 to A_3, followed by A_3 to A_1, and then finally she joins A_1 to A_0. This is illustrated in the figure (on the left) below.



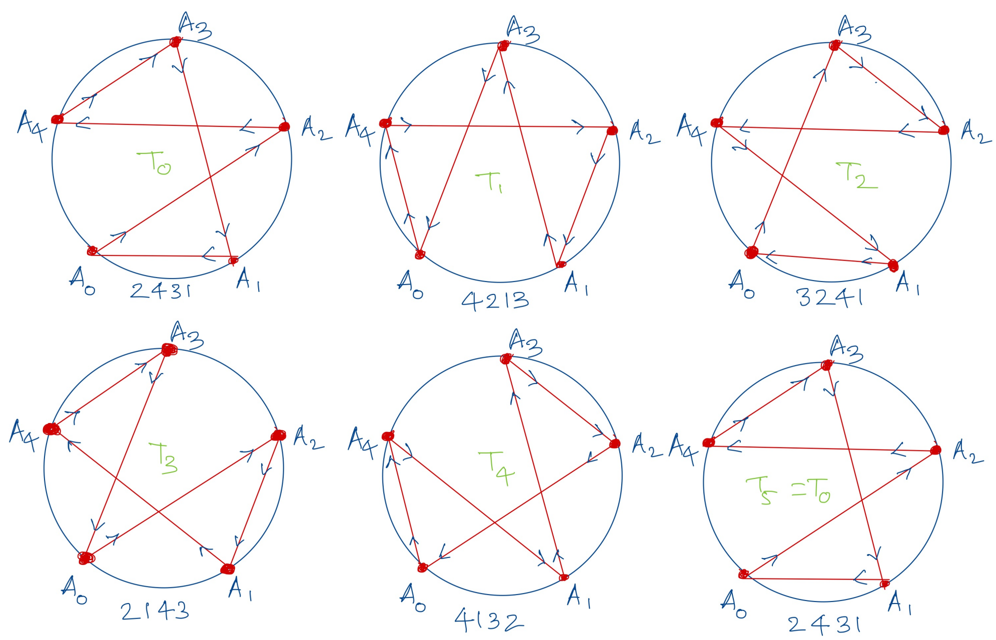
   It is clear that if we rotate (counter-clockwise) the figure by $(360°/5) = 72°$, we will get another closed figure (with possibly different shape and arrows directions), and hence another permutation of $\{1, 2, 3, 4\}$. This is shown in the picture above on the right.

   Show that each closed figure is one of the following two types: (i) **Type 1:** the closed figure will remain the same after one rotation by $72°$ (meaning that it will have the same shape and arrow directions after rotation)(ii) **Type 2:** the closed figure will come back to itself after 5 rotations (each of $72°$).

Show that in this case there are 4 different Type-1 figures. The picture below shows two such closed figures. Can you find the other two? Note that even if two closed figures have the same shape but different arrow directions, they are considered to be distinct.



(b) For a closed figure of Type 2 (call it T_0), by successively applying a rotation of $72°$, we get figures T_1, T_2, T_3, T_4 which are distinct from each other and from T_0. If we apply one more rotation of $72°$ to T_4, we will get back to T_0. This is also illustrated in the following picture.'



Can you find some more figures of Type 2?

(c) Now, using the same idea, for a given prime p, conclude that there are $(p-1)$ closed figures of Type 1. Show also that the set of closed figures of Type 2 can be divided exactly into groups of p distinct closed figures.

(d) Aakriti concludes that for each prime p, the number $(p-1)! + 1$ is divisible by p. Is she correct?

(e) Professor Wilson comes to visit Aakriti's school one day and brings with him what he calls a "magic formula." This formula takes two natural numbers as the input and the output, he claims, is always a prime number. Furthermore, he says that every prime number comes as the output of this formula for suitably chosen input. This is his magic formula:

$$F(x, y) = \left(\frac{y-1}{2}\right)\left(|A^2 - 1| - (A^2 - 1)\right) + 2,$$

where $A = x(y+1) - (y! + 1)$, and x and y are natural numbers (which are the input). Here, the notation $|A^2 - 1|$ means the absolute value of $(A^2 - 1)$.

Using Part (d) (or otherwise), can you verify Professor Wilson's claims?

Roundtable with friends

2. Naina, Payel, Rumi, Tina, Anjali and Diksha all study in the same class. However, not all of them are friends. In fact, each of them does not get along well at all with exactly two others and is friends with the remaining people. All of them are invited to a birthday party and asked to be seated around a circular table.

   (a) Will they all manage to sit comfortably ensuring that everyone sits between two friends?

   (b) Now, try to analyse a slightly more generalised situation as follows:
   Suppose $2n$ people are attending a party where each person does not get along well at all with at most $n-1$ others, and has friendly relations with the remaining people. Is it possible for all of them to sit comfortably around a circular table so that each person gets to sit between two people he/she is friends with?